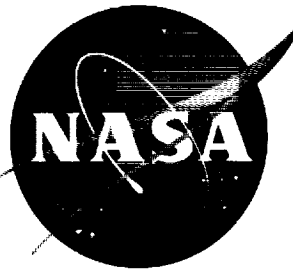


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## TECHNICAL NOTE

D-1604

EQUATIONS FOR DETERMINING VEHICLE POSITION IN EARTH-MOON  
SPACE FROM SIMULTANEOUS ONBOARD OPTICAL MEASUREMENTS

By Alton P. Mayo, Harold A. Hamer, and Margery E. Hannah

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### EQUATIONS FOR DETERMINING VEHICLE POSITION IN EARTH-MOON

### SPACE FROM SIMULTANEOUS ONBOARD OPTICAL MEASUREMENTS

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#### SUMMARY

Equations for 13 different combinations of sightings are presented for determining the position vector of a vehicle in earth-moon space from simultaneous onboard optical measurements. The methods for reducing the measurements for position determination are basically triangulation methods and incorporate combinations of measurements made on either one, two, or three bodies (earth, moon, and sun). Angular measurements made at the vehicle consist of declination of a body, right ascension of a body, body angular diameter, and included angles between a star and body center, star and horizon, moon center and earth horizon, landmark and star, landmark and body center, orbiting beacon and star, and orbiting beacon and body center. The methods presented apply to the determination of the total position vector (distance and direction from a reference body). With the exception of one method, the methods do not require the use of a stable platform.

The equations pertinent to the various methods are presented principally as background material for studies to determine practical and accurate methods for onboard optical navigation in earth-moon space.

#### INTRODUCTION

In the region between the earth and the moon, optical navigation techniques can be used to obtain a position fix whenever the closest body is out of range of onboard radar. This region where optical measurements or sightings may be most useful can be separated into three sections: a region near the earth where most of the optical sightings are made on the earth, an intermediate region where sightings are made on both the earth and moon, and a region near the moon where most of the optical sightings are made on the moon. It is conceivable that a lunar trip sightings made on the sun may also prove useful.

Numerous combinations of optical measurements can be made onboard the lunar space vehicle for navigational purposes. Each measurement generally includes a sighting to either the earth, moon, or sun (hereinafter referred to as bodies). Among the onboard measurements are body angular diameter, declination and right ascension of a body, and angular sightings between a star and a body center, between a star and a body horizon, between a star and a landmark or an orbiting

beacon, between a body center and a landmark or orbiting beacon, and between a body center and the horizon of another body. Radar-distance measurements made from a body and relayed to the vehicle are also of value for use in conjunction with the optical measurements.

The use of these types of measurements for determining the vehicle position vector in interplanetary space has been investigated in limited detail in references 1 and 2. The use of nonsimultaneous onboard optical measurements in a statistical process for navigational purposes has been investigated in references 3 and 4.

The present report is part of a study of onboard navigational systems which make use of certain combinations of simultaneous onboard optical measurements to obtain a position fix. The main purpose of this report is to present the pertinent equations for 13 selected combinations. Each combination contains the minimum number of simultaneous measurements for a nonredundant mathematical determination of the total vehicle position vector (distance and direction from a reference body). These 13 combinations were selected as a cross section of the much larger number of possible combinations and include sightings on either one, two, or three bodies (earth, moon, and sun).

The pertinent equations are presented as background material for studies to determine practical and accurate methods for onboard optical navigation in earth moon space. The equations will be especially useful for any detailed study, such as a complete error analysis of a particular system of optical measurements.

## SYMBOLS

i	inclination of orbit plane of beacon to earth equatorial plane
l,m,n	direction cosine of a line with respect to X-, Y-, and Z-axis, respectively
Q	orbital radius of orbiting beacon
R	radius of body
r	magnitude of position vector, $(x^2 + y^2 + z^2)^{1/2}$
t	time of observation (measured from reference time $t = 0$ )
X,Y,Z	rectangular right-hand axis system where X-axis is in the direction of Aries and Z-axis is in the direction of north celestial pole
x,y,z	position coordinates in rectangular right-hand axis system
$\alpha$	one-half of angular diameter of body as viewed from vehicle
$\theta$	angle formed at vehicle by two lines of sight

right ascension of landmark as measured from earth at  $t = 0$

geocentric latitude of landmark

angle in orbital plane measured eastward from ascending node to beacon  
at  $t = 0$

right ascension of ascending node of beacon orbit as measured in an  
earth-centered coordinate system (i.e., arc in earth's equatorial  
plane measured from the positive X-axis to intersection which orbital  
plane makes with equatorial plane as beacon passes from south to  
north)

angular rate of rotation of beacon eastward in its circular orbit

angular rate of rotation of earth about its axis

scripts:

orbiting beacon about earth

earth

earth center

earth horizon

landmark on earth

moon

moon center

sun

sun center

vehicle

1,3,4 star 1, star 2, star 3, and star 4

ation:

| rectangular matrix

} column matrix

absolute value

Bar over a symbol indicates a vector.

Two subscripts are used with position coordinates; for example,  $x_{ve}$ ,  $y_{ve}$ , and  $z_{ve}$  are the coordinates of the earth in a vehicle-centered system.

Two subscripts are used with angles except for body angular diameter; for example,  $\theta_{lm}$  is the angle included at the vehicle by star 1 and the moon center.

## DEVELOPMENT OF METHODS

### General Comments

The number of different optical navigational methods (combinations of sightings) available for earth-moon flight is large. An attempt has been made, however, to include sufficient combinations to give an indication of the range of possible measurements. Most of the navigational methods selected result in relatively simple expressions for the position vector.

Angular measurements made at the vehicle consist of declination and right ascension of a body, body angular diameter, and included angles between a star and body center, star and horizon, moon center and earth horizon, landmark and star, landmark and body center, orbiting beacon and star, and orbiting beacon and body center. Also incorporated in one of the methods is a radar-distance measurement made from a body and relayed to the vehicle. In any of the methods the solution for the vehicle position vector requires the knowledge of one or more distance values (such as earth, moon, or sun diameter, distance between earth and moon, etc.).

In each of the methods presented, a vehicle-centered right-hand rectangular system of axes was used in deriving the equations. In this system, the X and Y axes are oriented in a plane parallel to the earth equatorial plane with the X-axis in the direction of Aries and the Z-axis in the direction of the north celestial pole. None of the angular measurements requires a sign convention, except the declination and right-ascension measurements in method II. Also, this method is the only method requiring the use of a stable platform.

Each measurement will generally yield a surface in space, anywhere on which the vehicle may be located. For example, the angle measured between a star and a body center will fix the position of the vehicle on a cone with the body as the vertex. The different surfaces of position which are determined by the various measurements are illustrated in figures 1 to 3. Each of the navigational methods presented herein combine these surfaces in such a manner as to pinpoint the vehicle position with respect to a body. For any one method, the minimum number of measurements are used that will provide a nonredundant mathematical solution of the vehicle position vector.

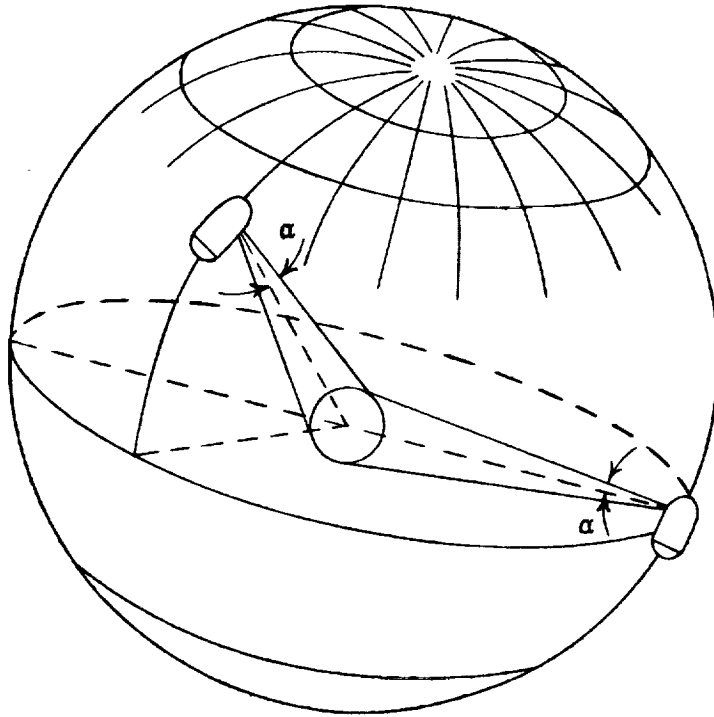


Figure 1.- Sphere of position of a space vehicle obtained by measuring angular diameter of a body or by measuring radar range from a body.

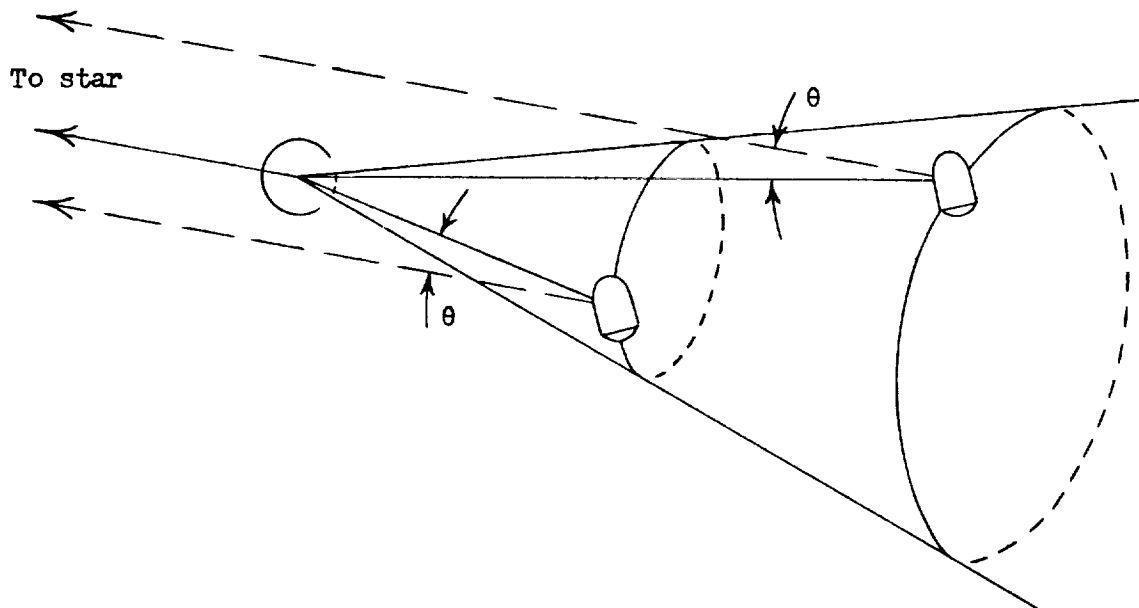
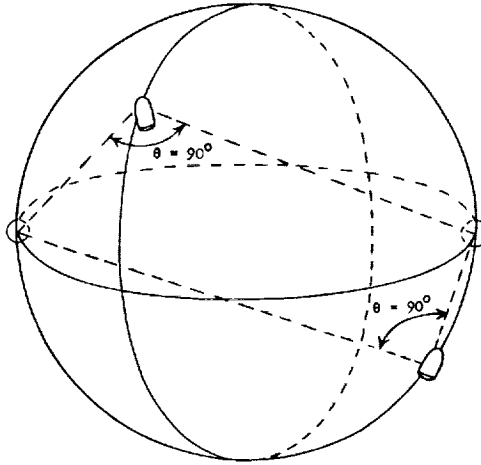
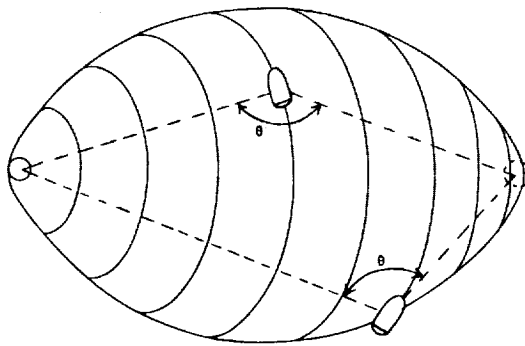


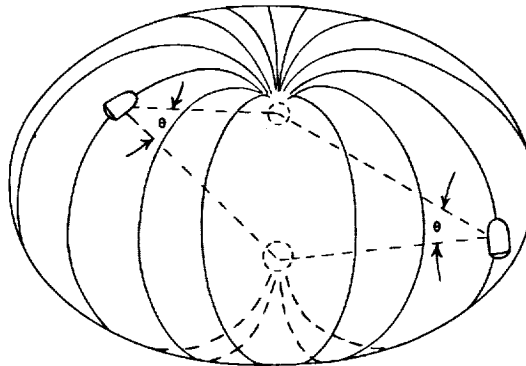
Figure 2.- Cone of position of a space vehicle obtained by measuring the angle included between a star and a point on or near a body (body center, body horizon, landmark, or orbiting beacon).



(a) Sphere of position: included angle =  $90^\circ$ .



(b) Included angle  $> 90^\circ$ .



(c) Included angle  $< 90^\circ$ .

Figure 3.- Surface of position of a space vehicle obtained by measuring the included angle between two bodies.

## Measurements

One-body (earth or moon).- Six methods involving measurements on one body (earth or moon) are presented. Four of these methods involve measurements of the angular diameter of the body. Of these four methods, one employs measurements of the angle included between a star and body center, one employs a stable platform, one makes use of sightings of a landmark, and one makes use of sightings of an orbiting beacon. Because angular-diameter measurements in general involve low rates of change with distance and are difficult to make accurately at close range, two methods which do not incorporate this measurement are included. One of these two methods makes use of a radar-distance measurement made from the earth and relayed to the vehicle. In the other method, measurements of the included angle between a star and the horizon are used since this measurement is rather sensitive to vehicle position change over the entire region of earth-moon space. This method which involves only one type of measurement avoids direct determination of body angular diameter and because of this may have desirable characteristics. The orbiting-beacon and landmark measurements are included because these measurements have high rates of change with time and may have effects on accuracy which are different from those of other types of measurements. The methods utilizing measurements on one body cover a wide variety of measurements and are illustrative of the results ordinarily obtainable from one-body measurements.

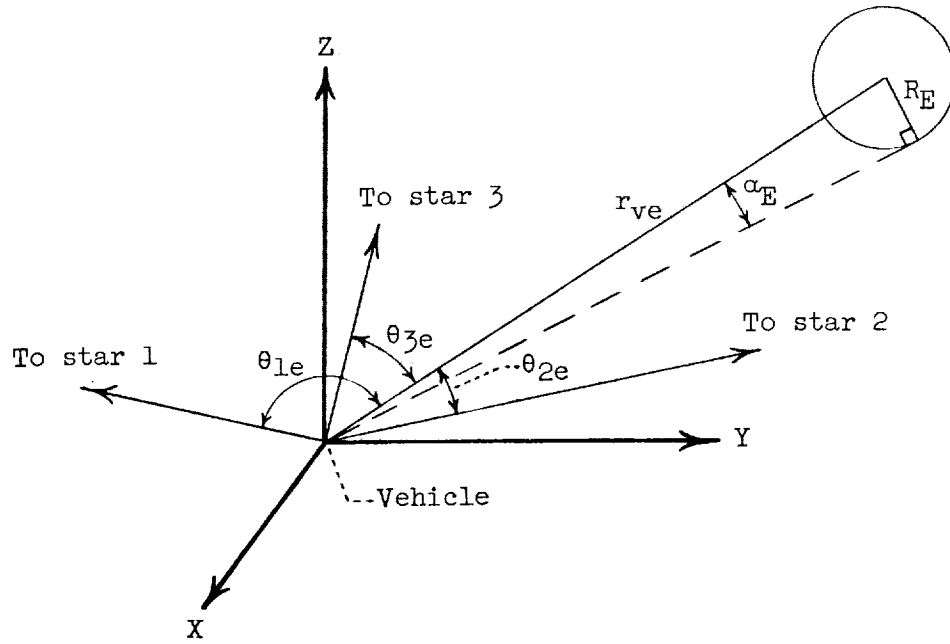
Two-body (earth and moon).- Three methods involving measurements on two bodies (earth and moon) were selected in such a manner that two methods utilize body-angular-diameter measurements and one does not. The measurements used in any one of the three methods were about equally divided between measurements made on the earth and measurements made on the moon. The results obtained from these methods are illustrative of those obtainable from any triangulation method involving both the earth and the moon.

Three-body (earth, moon, and sun).- The measurements on three bodies (earth, moon, and sun) represent the maximum number of bodies on which measurements will probably be made in earth-moon space. Of the four methods presented herein, one measures only directions (angles) between stars and body centers. The other three methods include a measurement of the angular diameter of either one, two, or three of the bodies. It is presently not known how effective the three-body methods may be. It is suspected, however, that the methods involving the angular diameter of the sun will have poor accuracy in the earth-moon region because this angle has a very small variation over this region of space.

## Derivation of Equations

Method I.- Measurements of included angles between each of three stars and a body center and the angular diameter of the body will yield the vehicle position vector. A sighting on a star and on a body center results in a cone of position with the vertex at the body center. (See fig. 2.) Sightings on any two stars will produce two cones which intersect, giving two possible lines of position. The third star (cone) establishes the actual line of position of the vehicle. The sphere of position obtained by the body-angular-diameter measurement (fig. 1) determines the location of the vehicle along this line of position. The

combination of optical angular measurements for method I is illustrated in sketch 1 (earth taken as an example).



Sketch 1

The cosine of the angle at the vehicle included by star 1 and the earth center is

$$\cos \theta_{1e} = l_1 l_e + m_1 m_e + n_1 n_e \quad (1)$$

but

$$\left. \begin{aligned} l_e &= \frac{x_{ve}}{r_{ve}} \\ m_e &= \frac{y_{ve}}{r_{ve}} \\ n_e &= \frac{z_{ve}}{r_{ve}} \end{aligned} \right\} \quad (2)$$

that

$$\cos \theta_{1e} = \frac{l_1 x_{ve} + m_1 y_{ve} + n_1 z_{ve}}{r_{ve}} \quad (3)$$

on consideration of the angular diameter of the earth,

$$r_{ve} = \frac{R_E}{\sin \alpha_E} \quad (4)$$

and substituting this expression into equation (3) gives

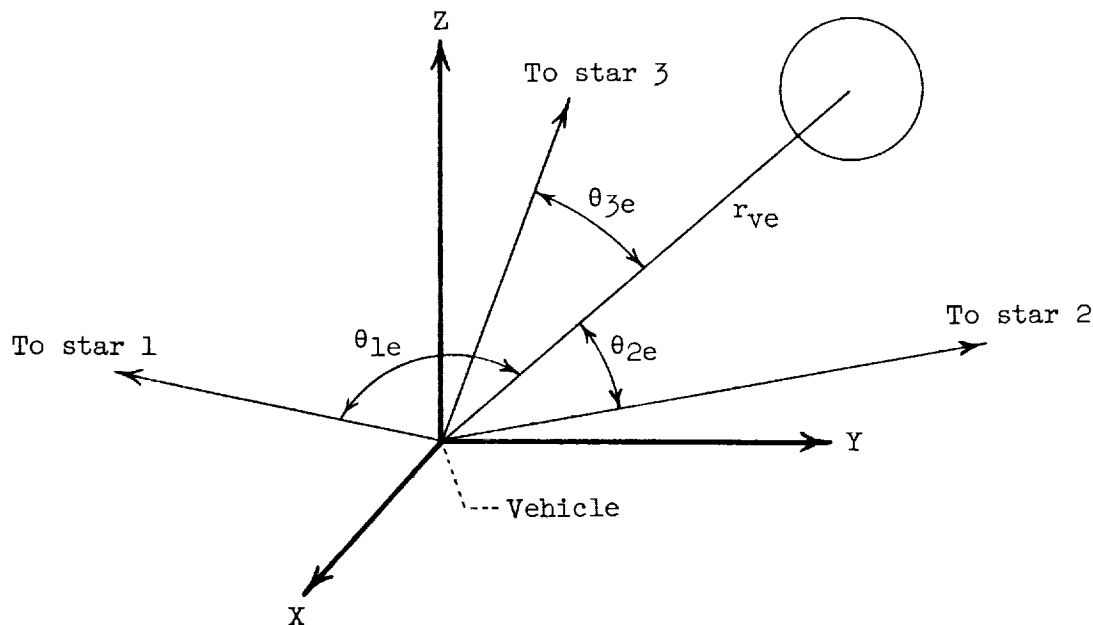
$$l_1 x_{ve} + m_1 y_{ve} + n_1 z_{ve} = \frac{R_E}{\sin \alpha_E} \cos \theta_{1e} \quad (5)$$

Expressing the relations for the two other stars in a similar manner results in the following set of equations for the determination of vehicle position by Method I:

$$\begin{Bmatrix} x_{ve} \\ y_{ve} \\ z_{ve} \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}^{-1} \begin{Bmatrix} \frac{R_E}{\sin \alpha_E} \cos \theta_{1e} \\ \frac{R_E}{\sin \alpha_E} \cos \theta_{2e} \\ \frac{R_E}{\sin \alpha_E} \cos \theta_{3e} \end{Bmatrix} \quad (6)$$

In this method, as in the other methods, it is assumed that the stars can be recognized and that their direction cosines are known.

Method II.— Method II is similar to method I but avoids a body angular-meter measurement. The angular-diameter measurement is replaced by a radar measurement, made from a body, of the distance between the vehicle and body center. This method is illustrated in sketch 2.

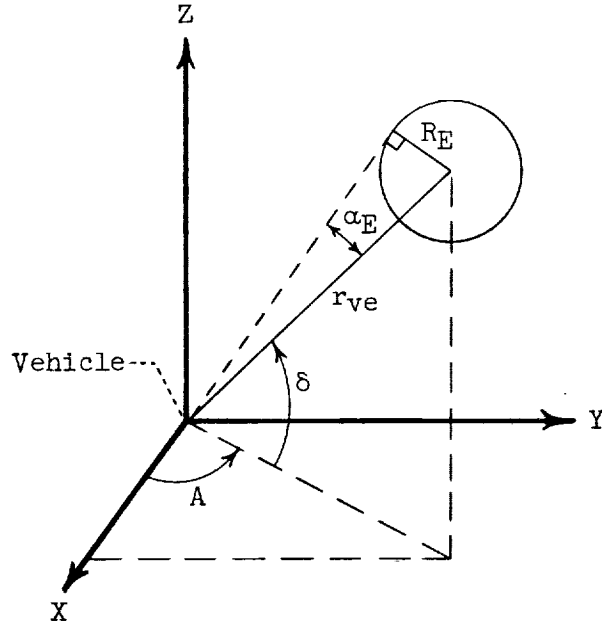


Sketch 2

From the procedure given for method I (see eqs. (1) to (3)), the following set of equations is obtained for determination of the vehicle position by method II:

$$\begin{Bmatrix} x_{ve} \\ y_{ve} \\ z_{ve} \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}^{-1} \begin{Bmatrix} r_{ve} \cos \theta_{1e} \\ r_{ve} \cos \theta_{2e} \\ r_{ve} \cos \theta_{3e} \end{Bmatrix} \quad (7)$$

Method III.— This method for determining the vehicle position vector involve measurements of the right ascension, declination, and angular diameter of a body. The right-ascension and declination sightings are made with reference to a stable platform. These two sightings determine a line of position in space and the body angular diameter determines the vehicle location along this line. The measurements for method III are illustrated in sketch 3; the platform is inertially stabilized and oriented parallel to the XYZ-axis system.



Sketch 3

From sketch 3, it is seen that

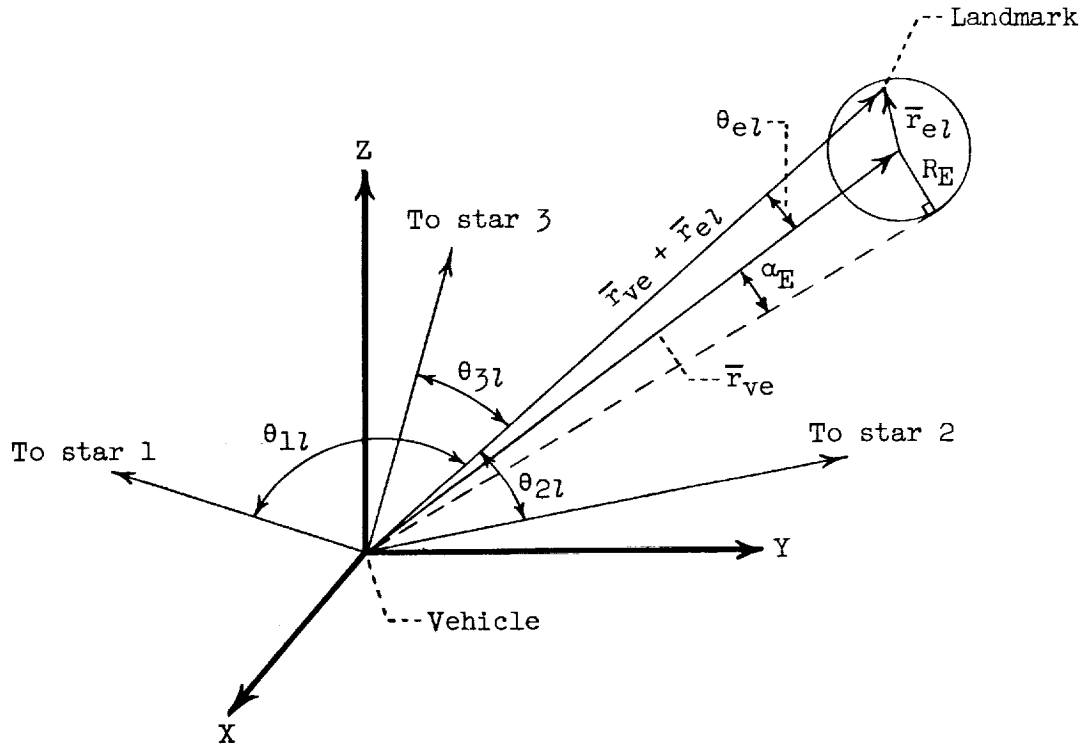
$$\left. \begin{aligned} x_{ve} &= r_{ve} \cos \delta \cos A \\ y_{ve} &= r_{ve} \cos \delta \sin A \\ z_{ve} &= r_{ve} \sin \delta \end{aligned} \right\} \quad (8)$$

where  $\delta$  is the declination of the earth (positive north from the X-Y plane of platform) and  $A$  is the right ascension of the earth as measured in the plane of the platform (positive eastward from the X-axis).

Substituting equation (4) into equations (8) gives the following set of equations to determine vehicle position by method III:

$$\begin{Bmatrix} x_{ve} \\ y_{ve} \\ z_{ve} \end{Bmatrix} = \begin{Bmatrix} \frac{R_E}{\sin \alpha_E} \cos \delta \cos A \\ \frac{R_E}{\sin \alpha_E} \cos \delta \sin A \\ \frac{R_E}{\sin \alpha_E} \sin \delta \end{Bmatrix} \quad (9)$$

Method IV.- In method IV, vehicle position is determined by measurements of included angles between each of three stars and a landmark, included angle between the body center and landmark, and angular diameter of the body. The various sightings are illustrated in sketch 4.



Sketch 4

In sketch 4, the position vector of the center of the earth as measured from the vehicle is

$$\bar{r}_{ve} = \bar{x}_{ve} + \bar{y}_{ve} + \bar{z}_{ve} \quad (10)$$

and the position vector of the landmark with origin at the center of the earth is

$$\bar{r}_{el} = \bar{x}_{el} + \bar{y}_{el} + \bar{z}_{el} \quad (11)$$

The distance from the vehicle to the landmark as determined from equations (10) and (11) is

$$|\bar{r}_{vl}| = |\bar{r}_{ve} + \bar{r}_{el}| = \left( |\bar{x}_{ve} + \bar{x}_{el}|^2 + |\bar{y}_{ve} + \bar{y}_{el}|^2 + |\bar{z}_{ve} + \bar{z}_{el}|^2 \right)^{1/2} \quad (12)$$

By reasoning similar to that employed in method I, the cosines of the angles included at the vehicle by stars 1, 2, and 3 and the landmark, and by the center of the earth and the landmark, respectively, may be determined from the following expressions:

$$r_{vl} \cos \theta_{1l} = l_1(x_{ve} + x_{el}) + m_1(y_{ve} + y_{el}) + n_1(z_{ve} + z_{el}) \quad (13a)$$

$$r_{vl} \cos \theta_{2l} = l_2(x_{ve} + x_{el}) + m_2(y_{ve} + y_{el}) + n_2(z_{ve} + z_{el}) \quad (13b)$$

$$r_{vl} \cos \theta_{3l} = l_3(x_{ve} + x_{el}) + m_3(y_{ve} + y_{el}) + n_3(z_{ve} + z_{el}) \quad (13c)$$

$$r_{vl} \cos \theta_{el} = \frac{x_{ve}(x_{ve} + x_{el}) + y_{ve}(y_{ve} + y_{el}) + z_{ve}(z_{ve} + z_{el})}{(x_{ve}^2 + y_{ve}^2 + z_{ve}^2)^{1/2}} \quad (13d)$$

Multiplying equation (13b) by  $\frac{\cos \theta_{1l}}{\cos \theta_{2l}}$  and subtracting the result from equation (13a) yields

$$\begin{aligned} & \left(1 - l_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}}\right)x_{ve} + \left(m_1 - m_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}}\right)y_{ve} + \left(n_1 - n_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}}\right)z_{ve} \\ &= \left(l_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}} - l_1\right)x_{el} + \left(m_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}} - m_1\right)y_{el} + \left(n_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}} - n_1\right)z_{el} \end{aligned} \quad (14)$$

using a similar procedure between equations (13c) and (13a) and between equations (13d) and (13a), and substituting

$$(x_{ve}^2 + y_{ve}^2 + z_{ve}^2)^{1/2} = r_{ve} = \frac{R_E}{\sin \alpha_E}$$

into the last of the resulting equations, the following set of equations is obtained for method IV:

$$\begin{Bmatrix} x_{ve} \\ y_{ve} \\ z_{ve} \end{Bmatrix} = \begin{bmatrix} l_1 - l_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}} & m_1 - m_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}} & n_1 - n_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}} \\ l_1 - l_3 \frac{\cos \theta_{1l}}{\cos \theta_{3l}} & m_1 - m_3 \frac{\cos \theta_{1l}}{\cos \theta_{3l}} & n_1 - n_3 \frac{\cos \theta_{1l}}{\cos \theta_{3l}} \\ l_1 - x_{el} \frac{\cos \theta_{1l} \sin \alpha_E}{R_E \cos \theta_{el}} & m_1 - y_{el} \frac{\cos \theta_{1l} \sin \alpha_E}{R_E \cos \theta_{el}} & n_1 - z_{el} \frac{\cos \theta_{1l} \sin \alpha_E}{R_E \cos \theta_{el}} \end{bmatrix}$$

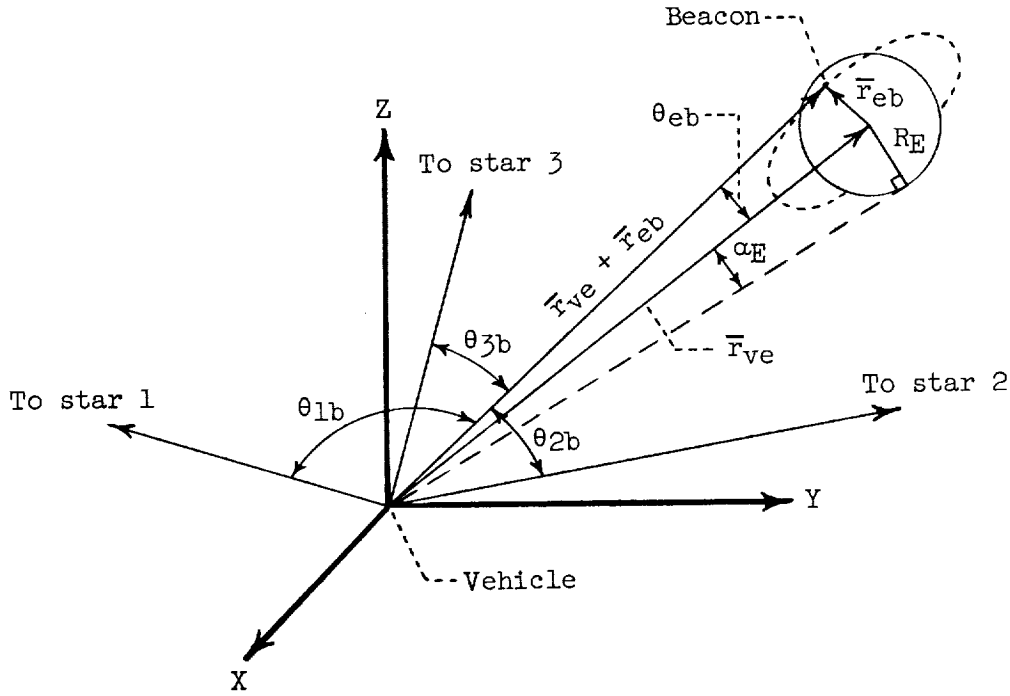
$$\times \begin{Bmatrix} \left( l_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}} - l_1 \right) x_{el} + \left( m_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}} - m_1 \right) y_{el} + \left( n_2 \frac{\cos \theta_{1l}}{\cos \theta_{2l}} - n_1 \right) z_{el} \\ \left( l_3 \frac{\cos \theta_{1l}}{\cos \theta_{3l}} - l_1 \right) x_{el} + \left( m_3 \frac{\cos \theta_{1l}}{\cos \theta_{3l}} - m_1 \right) y_{el} + \left( n_3 \frac{\cos \theta_{1l}}{\cos \theta_{3l}} - n_1 \right) z_{el} \\ R_E \left( \frac{\cos \theta_{1l}}{\sin \alpha_E \cos \theta_{el}} \right) - (l_1 x_{el} + m_1 y_{el} + n_1 z_{el}) \end{Bmatrix}$$

(15)

where

$$\left. \begin{aligned} x_{el} &= R_E \cos \varphi \cos(\omega_E t + \lambda_0) \\ y_{el} &= R_E \cos \varphi \sin(\omega_E t + \lambda_0) \\ z_{el} &= R_E \sin \varphi \end{aligned} \right\} \quad (16)$$

Method V.—Method V is similar to method IV except the landmark is replaced by a beacon in a circular orbit as illustrated in sketch 5.



Sketch 5

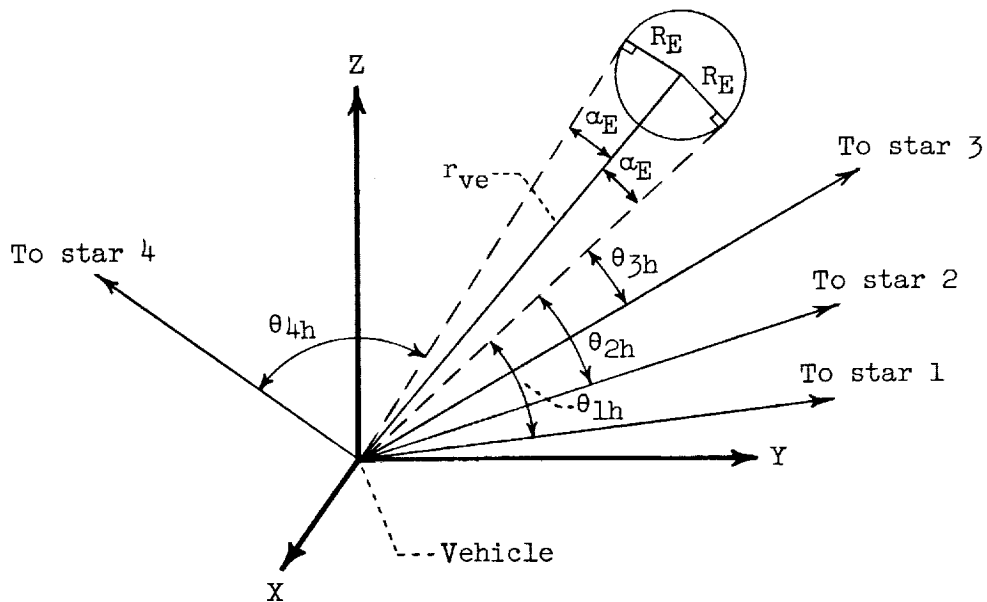
The equations are derived in the same manner as those for method IV and may be obtained by substituting angles to a beacon for angles to a landmark and defining  $x_{eb}$ ,  $y_{eb}$ , and  $z_{eb}$ . The set of equations for method V is as follows:

$$\begin{Bmatrix} x_{ve} \\ y_{ve} \\ z_{ve} \end{Bmatrix} = \begin{bmatrix} l_1 - l_2 \frac{\cos \theta_{1b}}{\cos \theta_{2b}} & m_1 - m_2 \frac{\cos \theta_{1b}}{\cos \theta_{2b}} & n_1 - n_2 \frac{\cos \theta_{1b}}{\cos \theta_{2b}} \\ l_1 - l_3 \frac{\cos \theta_{1b}}{\cos \theta_{3b}} & m_1 - m_3 \frac{\cos \theta_{1b}}{\cos \theta_{3b}} & n_1 - n_3 \frac{\cos \theta_{1b}}{\cos \theta_{3b}} \\ l_1 - x_{eb} \frac{\cos \theta_{1b} \sin \alpha_E}{R_E \cos \theta_{eb}} & m_1 - y_{eb} \frac{\cos \theta_{1b} \sin \alpha_E}{R_E \cos \theta_{eb}} & n_1 - z_{eb} \frac{\cos \theta_{1b} \sin \alpha_E}{R_E \cos \theta_{eb}} \end{bmatrix}^{-1} \times \begin{Bmatrix} \left( l_2 \frac{\cos \theta_{1b}}{\cos \theta_{2b}} - l_1 \right) x_{eb} + \left( m_2 \frac{\cos \theta_{1b}}{\cos \theta_{2b}} - m_1 \right) y_{eb} + \left( n_2 \frac{\cos \theta_{1b}}{\cos \theta_{2b}} - n_1 \right) z_{eb} \\ \left( l_3 \frac{\cos \theta_{1b}}{\cos \theta_{3b}} - l_1 \right) x_{eb} + \left( m_3 \frac{\cos \theta_{1b}}{\cos \theta_{3b}} - m_1 \right) y_{eb} + \left( n_3 \frac{\cos \theta_{1b}}{\cos \theta_{3b}} - n_1 \right) z_{eb} \\ R_E \left( \frac{\cos \theta_{1b}}{\sin \alpha_E \cos \theta_{eb}} \right) - (l_1 x_{eb} + m_1 y_{eb} + n_1 z_{eb}) \end{Bmatrix} \quad (17)$$

where

$$\left. \begin{aligned} x_{eb} &= Q \cos \left\{ \sin^{-1} \left[ \sin i \sin(\omega_b t + \psi) \right] \right\} \cos \left\{ \Omega + \tan^{-1} \left[ \cos i \tan(\omega_b t + \psi) \right] \right\} \\ y_{eb} &= Q \cos \left\{ \sin^{-1} \left[ \sin i \sin(\omega_b t + \psi) \right] \right\} \sin \left\{ \Omega + \tan^{-1} \left[ \cos i \tan(\omega_b t + \psi) \right] \right\} \\ z_{eb} &= Q \sin i \sin(\omega_b t + \psi) \end{aligned} \right\} \quad (18)$$

Method VI.— This method for determining vehicle position involves measurements of included angles between each of four stars and the horizon of a body. Method VI avoids the direct measurement of body angular diameter, although, as shown in sketch 6, this angle is used to establish the equations.



Sketch 6

For simplicity in drawing the sketch, the stars are shown as coplanar. This restriction, however, does not apply to the development of the equations. In fact, the stars must not be coplanar.

The angle included at the vehicle by the line of sight to star 1 and the line sight to the point on the earth horizon nearest to this star is

$$\theta_{1h} = \theta_{1e} - \alpha_E$$

en,

$$\theta_{1e} = \theta_{1h} + \alpha_E \quad (19)$$

the equation for the cosine of the angle included by star 1 and the earth center

$$\cos(\theta_{1h} + \alpha_E) = \frac{l_{1x_{ve}} + m_{1y_{ve}} + n_{1z_{ve}}}{r_{ve}} \quad (20)$$

, from substitution of equation (4) into equation (20),

$$\cos(\theta_{1h} + \alpha_E) \frac{R_E}{\sin \alpha_E} = l_{1x_{ve}} + m_{1y_{ve}} + n_{1z_{ve}} \quad (21)$$

After substituting the trigonometric function for  $\cos(\theta_{1h} + \alpha_E)$  and collecting terms, the following expression results:

$$\cot \alpha_E = \frac{1}{R_E \cos \theta_{1h}} (l_{1x_{ve}} + m_{1y_{ve}} + n_{1z_{ve}} + R_E \sin \theta_{1h}) \quad (22a)$$

Similar operations yield the following equations for the other three star-horizon measurements:

$$\cot \alpha_E = \frac{1}{R_E \cos \theta_{2h}} (l_{2x_{ve}} + m_{2y_{ve}} + n_{2z_{ve}} + R_E \sin \theta_{2h}) \quad (22b)$$

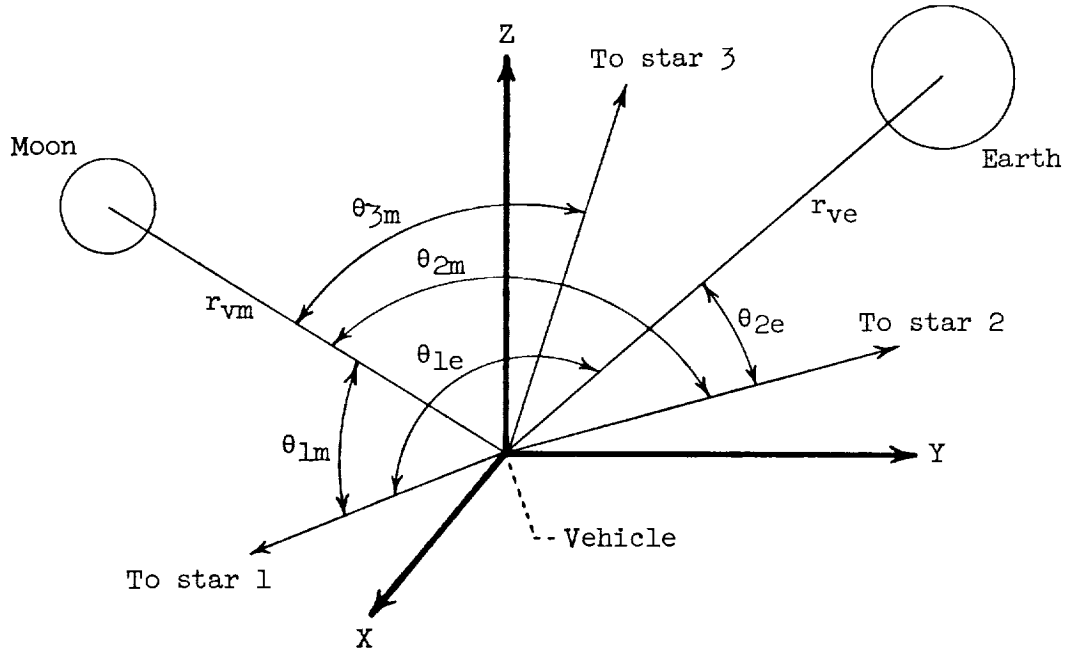
$$\cot \alpha_E = \frac{1}{R_E \cos \theta_{3h}} (l_{3x_{ve}} + m_{3y_{ve}} + n_{3z_{ve}} + R_E \sin \theta_{3h}) \quad (22c)$$

$$\cot \alpha_E = \frac{1}{R_E \cos \theta_{4h}} (l_4 x_{ve} + m_4 y_{ve} + n_4 z_{ve} + R_E \sin \theta_{4h}) \quad (22d)$$

Eliminating  $\cot \alpha_E$  from the four equations yields the following solution for the vehicle position by method VI:

$$\begin{Bmatrix} x_{ve} \\ y_{ve} \\ z_{ve} \end{Bmatrix} = R_E \begin{bmatrix} \frac{l_1}{\cos \theta_{1h}} - \frac{l_2}{\cos \theta_{2h}} & \frac{m_1}{\cos \theta_{1h}} - \frac{m_2}{\cos \theta_{2h}} & \frac{n_1}{\cos \theta_{1h}} - \frac{n_2}{\cos \theta_{2h}} \\ \frac{l_1}{\cos \theta_{1h}} - \frac{l_3}{\cos \theta_{3h}} & \frac{m_1}{\cos \theta_{1h}} - \frac{m_3}{\cos \theta_{3h}} & \frac{n_1}{\cos \theta_{1h}} - \frac{n_3}{\cos \theta_{3h}} \\ \frac{l_1}{\cos \theta_{1h}} - \frac{l_4}{\cos \theta_{4h}} & \frac{m_1}{\cos \theta_{1h}} - \frac{m_4}{\cos \theta_{4h}} & \frac{n_1}{\cos \theta_{1h}} - \frac{n_4}{\cos \theta_{4h}} \end{bmatrix}^{-1} \times \begin{Bmatrix} \frac{\sin \theta_{2h}}{\cos \theta_{2h}} - \frac{\sin \theta_{1h}}{\cos \theta_{1h}} \\ \frac{\sin \theta_{3h}}{\cos \theta_{3h}} - \frac{\sin \theta_{1h}}{\cos \theta_{1h}} \\ \frac{\sin \theta_{4h}}{\cos \theta_{4h}} - \frac{\sin \theta_{1h}}{\cos \theta_{1h}} \end{Bmatrix} \quad (23)$$

Method VII.- In method VII, vehicle position is determined by measurements made on two bodies. These measurements, illustrated in sketch 7, are the included angles between each of two stars and the earth center and between each of three stars and the moon center.



Sketch 7

For any method involving measurements on the earth and moon, the coordinates of the moon with respect to the earth center are required. This information is readily available from published ephemeris data.

The expressions for the cosines of the angles included at the vehicle by stars 1 and 2 and the earth center and by stars 1, 2, and 3 and the moon center are, respectively,

$$r_{ve} \cos \theta_{1e} = l_1 x_{ve} + m_1 y_{ve} + n_1 z_{ve} \quad (24a)$$

$$r_{ve} \cos \theta_{2e} = l_2 x_{ve} + m_2 y_{ve} + n_2 z_{ve} \quad (24b)$$

$$r_{vm} \cos \theta_{1m} = l_1 x_{vm} + m_1 y_{vm} + n_1 z_{vm} \quad (24c)$$

$$r_{vm} \cos \theta_{2m} = l_2 x_{vm} + m_2 y_{vm} + n_2 z_{vm} \quad (24d)$$

$$r_{vm} \cos \theta_{3m} = l_3 x_{vm} + m_3 y_{vm} + n_3 z_{vm} \quad (24e)$$

Then, multiplying equation (24b) by  $\frac{\cos \theta_{1e}}{\cos \theta_{2e}}$  and subtracting the result from equation (24a), multiplying equation (24c) by  $\frac{\cos \theta_{2m}}{\cos \theta_{1m}}$  and subtracting the result from equation (24d), multiplying equation (24e) by  $\frac{\cos \theta_{2m}}{\cos \theta_{3m}}$  and subtracting the result from equation (24d), and substituting

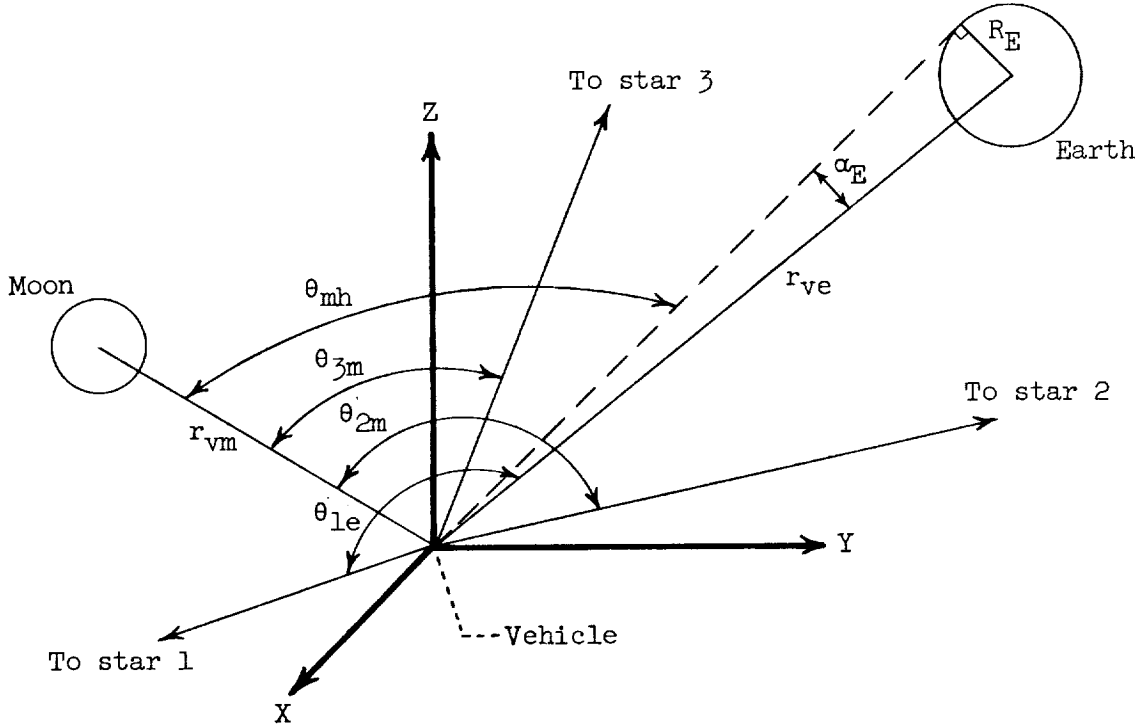
$$\left. \begin{aligned} x_{vm} &= x_{ve} + x_{em} \\ y_{vm} &= y_{ve} + y_{em} \\ z_{vm} &= z_{ve} + z_{em} \end{aligned} \right\} \quad (25)$$

into the resulting equations yields the following solution for the vehicle position by method VII:

$$\begin{Bmatrix} x_{ve} \\ y_{ve} \\ z_{ve} \end{Bmatrix} = \begin{bmatrix} l_1 - l_2 \frac{\cos \theta_{1e}}{\cos \theta_{2e}} & m_1 - m_2 \frac{\cos \theta_{1e}}{\cos \theta_{2e}} & n_1 - n_2 \frac{\cos \theta_{1e}}{\cos \theta_{2e}} \\ l_2 - l_1 \frac{\cos \theta_{2m}}{\cos \theta_{1m}} & m_2 - m_1 \frac{\cos \theta_{2m}}{\cos \theta_{1m}} & n_2 - n_1 \frac{\cos \theta_{2m}}{\cos \theta_{1m}} \\ l_2 - l_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} & m_2 - m_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} & n_2 - n_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} \end{bmatrix}^{-1} \times \begin{Bmatrix} 0 \\ \left( l_1 \frac{\cos \theta_{2m}}{\cos \theta_{1m}} - l_2 \right) x_{em} + \left( m_1 \frac{\cos \theta_{2m}}{\cos \theta_{1m}} - m_2 \right) y_{em} + \left( n_1 \frac{\cos \theta_{2m}}{\cos \theta_{1m}} - n_2 \right) z_{em} \\ \left( l_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} - l_2 \right) x_{em} + \left( m_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} - m_2 \right) y_{em} + \left( n_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} - n_2 \right) z_{em} \end{Bmatrix} \quad (26)$$

Method VIII.— Method VIII incorporates measurements of included angles between a star and the earth center, between each of two stars and the moon

center, between the moon center and earth horizon, and the angular diameter of the earth. The various sightings are illustrated in sketch 8.



Sketch 8

The expressions for cosines of the angles included at the vehicle by star 1 and the earth center, by stars 2 and 3 and the moon center, and by the moon center and earth horizon are, respectively,

$$R_E \frac{\cos \theta_{1e}}{\sin \alpha_E} = l_1 x_{ve} + m_1 y_{ve} + n_1 z_{ve} \quad (27a)$$

$$r_{vm} \cos \theta_{2m} = l_2 x_{vm} + m_2 y_{vm} + n_2 z_{vm} \quad (27b)$$

$$r_{vm} \cos \theta_{3m} = l_3 x_{vm} + m_3 y_{vm} + n_3 z_{vm} \quad (27c)$$

$$\frac{R_E}{\sin \alpha_E} r_{vm} \cos(\theta_{mh} + \alpha_E) = x_{vm}x_{ve} + y_{vm}y_{ve} + z_{vm}z_{ve} \quad (27d)$$

Elimination of  $r_{vm}$  between equations (27b) and (27c) and substitution of equations (25) into the resulting equation yields the second equation of matrix equation (28). The third equation of the matrix is obtained by multiplying equation (27b) by  $\frac{R_E}{\sin \alpha_E} \frac{\cos(\theta_{mh} + \alpha_E)}{\cos \theta_{2m}}$ , subtracting the result from equation (27d),

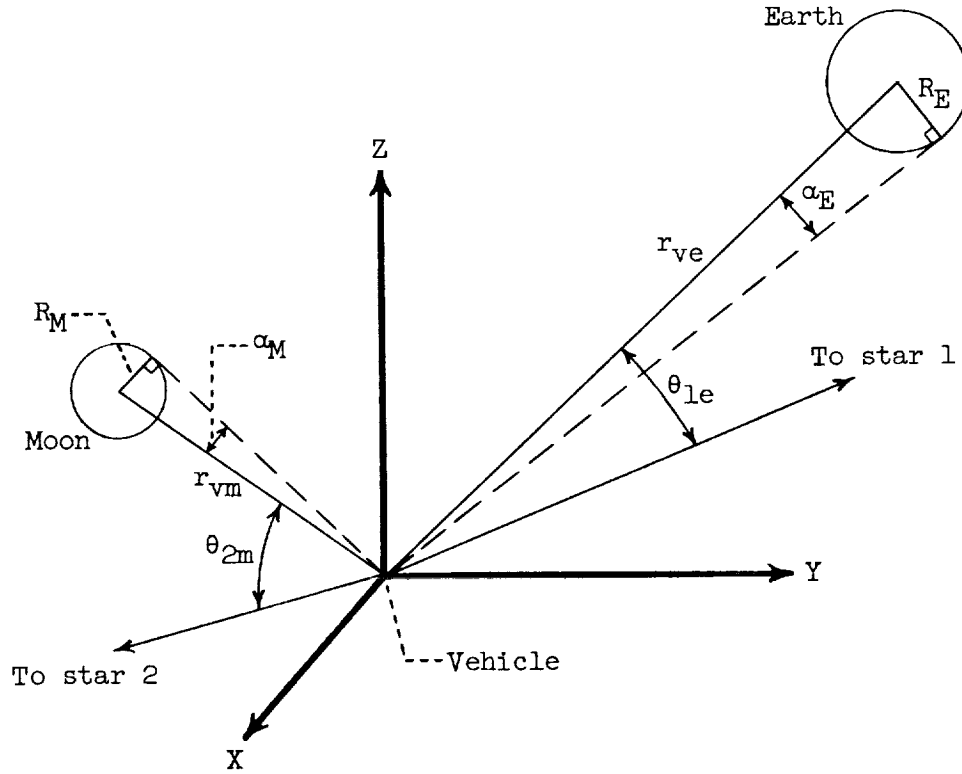
and substituting equations (25) and  $\frac{R_E^2}{\sin^2 \alpha_E} = x_{ve}^2 + y_{ve}^2 + z_{ve}^2$  into the

resulting equation. Hence, the determination of vehicle position by method VIII, as given by equation (28), is

$$\begin{Bmatrix} x_{ve} \\ y_{ve} \\ z_{ve} \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 - l_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} & m_2 - m_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} & n_2 - n_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} \\ x_{em} - l_2 \frac{R_E \cos(\theta_{mh} + \alpha_E)}{\sin \alpha_E \cos \theta_{2m}} & y_{em} - m_2 \frac{R_E \cos(\theta_{mh} + \alpha_E)}{\sin \alpha_E \cos \theta_{2m}} & z_{em} - n_2 \frac{R_E \cos(\theta_{mh} + \alpha_E)}{\sin \alpha_E \cos \theta_{2m}} \end{bmatrix}^{-1} \times \begin{Bmatrix} \frac{R_E}{\sin \alpha_E} \cos \theta_{1e} \\ \left( l_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} - l_2 \right) x_{em} + \left( m_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} - m_2 \right) y_{em} + \left( n_3 \frac{\cos \theta_{2m}}{\cos \theta_{3m}} - n_2 \right) z_{em} \\ \left( l_2 x_{em} + m_2 y_{em} + n_2 z_{em} \right) \frac{R_E}{\sin \alpha_E} \frac{\cos(\theta_{mh} + \alpha_E)}{\cos \theta_{2m}} - \frac{R_E^2}{\sin^2 \alpha_E} \end{Bmatrix}$$

(28)

Method IX.— Method IX involves measurements of the included angles between star and the earth center, between another star and the moon center, and the polar diameters of the earth and the moon. (See sketch 9.)



Sketch 9

cosines of the angles included at the vehicle by star 1 and the earth center by star 2 and the moon center are, respectively,

$$\frac{R_E}{\sin \alpha_E} \cos \theta_{1e} = l_1 x_{ve} + m_1 y_{ve} + n_1 z_{ve} \quad (29a)$$

$$\frac{R_M}{\sin \alpha_M} \cos \theta_{2m} = l_2 x_{vm} + m_2 y_{vm} + n_2 z_{vm} \quad (29b)$$

distances from the vehicle to the earth and from the vehicle to the moon are, respectively,

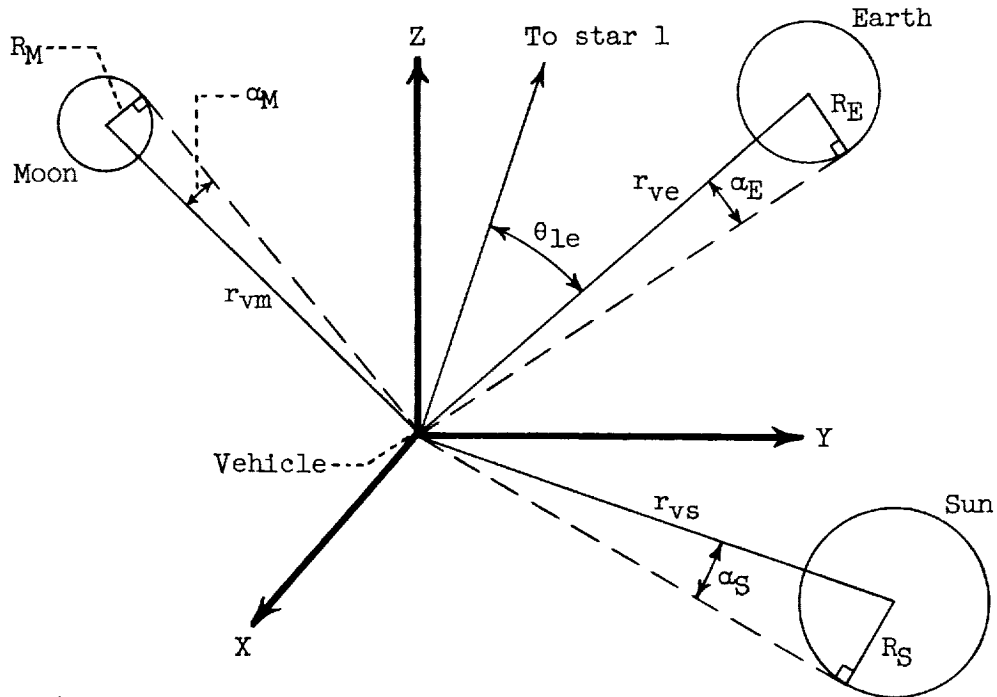
$$\frac{R_E}{\sin \alpha_E} = (x_{ve}^2 + y_{ve}^2 + z_{ve}^2)^{1/2} \quad (30a)$$

$$\frac{R_M}{\sin \alpha_M} = (x_{vm}^2 + y_{vm}^2 + z_{vm}^2)^{1/2} \quad (30b)$$

Squaring equations (30a) and (30b), subtracting (30a) from (30b), and substituting equations (25) into the resulting equation and into equation (29b) yields the second and third equations in the following matrix equation for determination of vehicle position by method IX:

$$\begin{Bmatrix} x_{ve} \\ y_{ve} \\ z_{ve} \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ 2x_{em} & 2y_{em} & 2z_{em} \end{bmatrix}^{-1} \begin{Bmatrix} R_E \frac{\cos \theta_{1e}}{\sin \alpha_E} \\ R_M \frac{\cos \theta_{2m}}{\sin \alpha_M} - l_2 x_{em} - m_2 y_{em} - n_2 z_{em} \\ \frac{R_M^2}{\sin^2 \alpha_M} - \frac{R_E^2}{\sin^2 \alpha_E} - r_{em}^2 \end{Bmatrix} \quad (31)$$

Method X.— In method X, measurements are made of the included angle between a star and the earth center and the angular diameters of the earth, moon, and sun. (See sketch 10.)



Sketch 10

For any method involving measurements on the earth, moon, and sun, the coordinates of the moon and sun with respect to the earth center are required (ephemeris data).

The angle included at the vehicle by the star and the earth center may be found from the following equation:

$$R_E \frac{\cos \theta_{le}}{\sin \alpha_E} = l_1 x_{ve} + m_1 y_{ve} + n_1 z_{ve} \quad (32)$$

The squares of the distances of the earth, moon, and sun from the vehicle are, respectively,

$$\frac{R_E^2}{\sin^2 \alpha_E} = x_{ve}^2 + y_{ve}^2 + z_{ve}^2 \quad (33a)$$

$$\frac{R_M^2}{\sin^2 \alpha_M} = x_{vm}^2 + y_{vm}^2 + z_{vm}^2 \quad (33b)$$

$$\frac{R_S^2}{\sin^2 \alpha_S} = x_{vs}^2 + y_{vs}^2 + z_{vs}^2 \quad (33c)$$

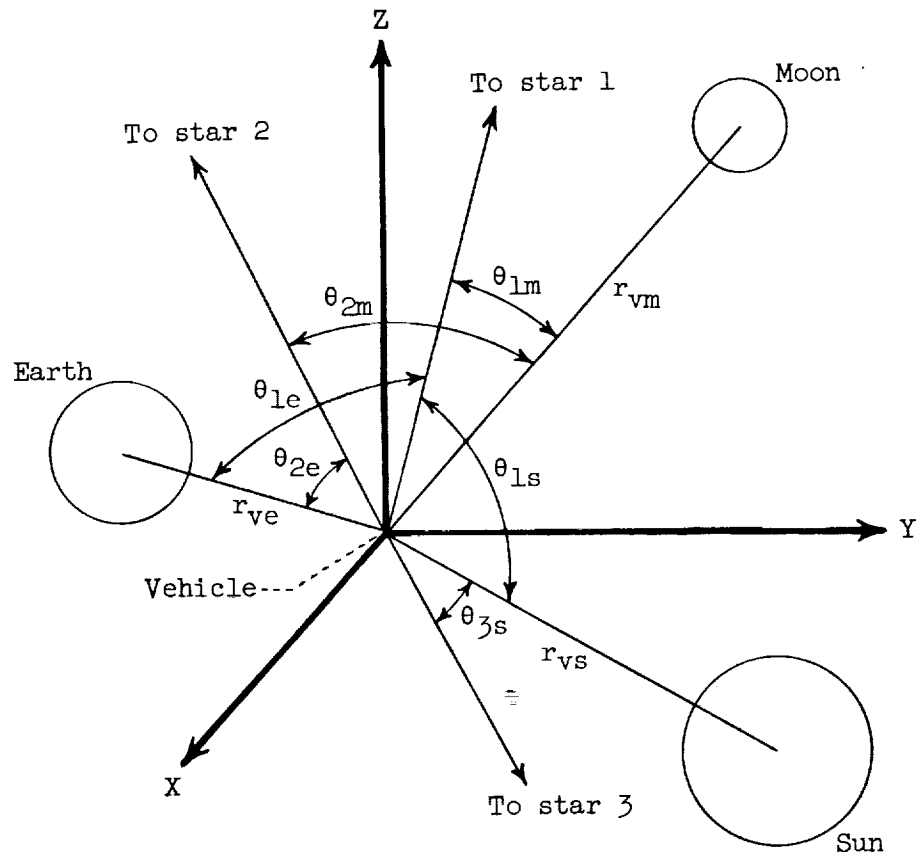
Substitution of equation (32) with the two equations obtained by subtracting equation (33a) from equation (33b) and equation (33a) from equation (33c) and substitution of

$$\left. \begin{aligned} x_{vm} &= x_{ve} + x_{em} \\ x_{vs} &= x_{ve} + x_{es} \\ y_{vm} &= y_{ve} + y_{em} \\ y_{vs} &= y_{ve} + y_{es} \\ z_{vm} &= z_{ve} + z_{em} \\ z_{vs} &= z_{ve} + z_{es} \end{aligned} \right\} \quad (34)$$

into the result yields the following solution by method X:

$$\begin{Bmatrix} x_{ve} \\ y_{ve} \\ z_{ve} \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ 2x_{em} & 2y_{em} & 2z_{em} \\ 2x_{es} & 2y_{es} & 2z_{es} \end{bmatrix}^{-1} \begin{Bmatrix} R_E \frac{\cos \theta_{1e}}{\sin \alpha_E} \\ \frac{R_M^2}{\sin^2 \alpha_M} - \frac{R_E^2}{\sin^2 \alpha_E} - r_{em}^2 \\ \frac{R_S^2}{\sin^2 \alpha_S} - \frac{R_E^2}{\sin^2 \alpha_E} - r_{es}^2 \end{Bmatrix} \quad (3)$$

Method XI.— Method XI incorporates measurements of included angles between three stars and the centers of the earth, moon, and sun. As illustrated in sketch 11, not all of the stars are used in connection with all of the bodies.



Sketch 11

the angles included between star 1 and the centers of the earth, moon, and sun, between star 2 and the centers of the earth and moon, and between star 3 and the center of the sun may be found from, respectively,

$$r_{ve} \cos \theta_{1e} = l_1 x_{ve} + m_1 y_{ve} + n_1 z_{ve} \quad (36a)$$

$$r_{vm} \cos \theta_{1m} = l_1 x_{vm} + m_1 y_{vm} + n_1 z_{vm} \quad (36b)$$

$$r_{vs} \cos \theta_{1s} = l_1 x_{vs} + m_1 y_{vs} + n_1 z_{vs} \quad (36c)$$

$$r_{ve} \cos \theta_{2e} = l_2 x_{ve} + m_2 y_{ve} + n_2 z_{ve} \quad (36d)$$

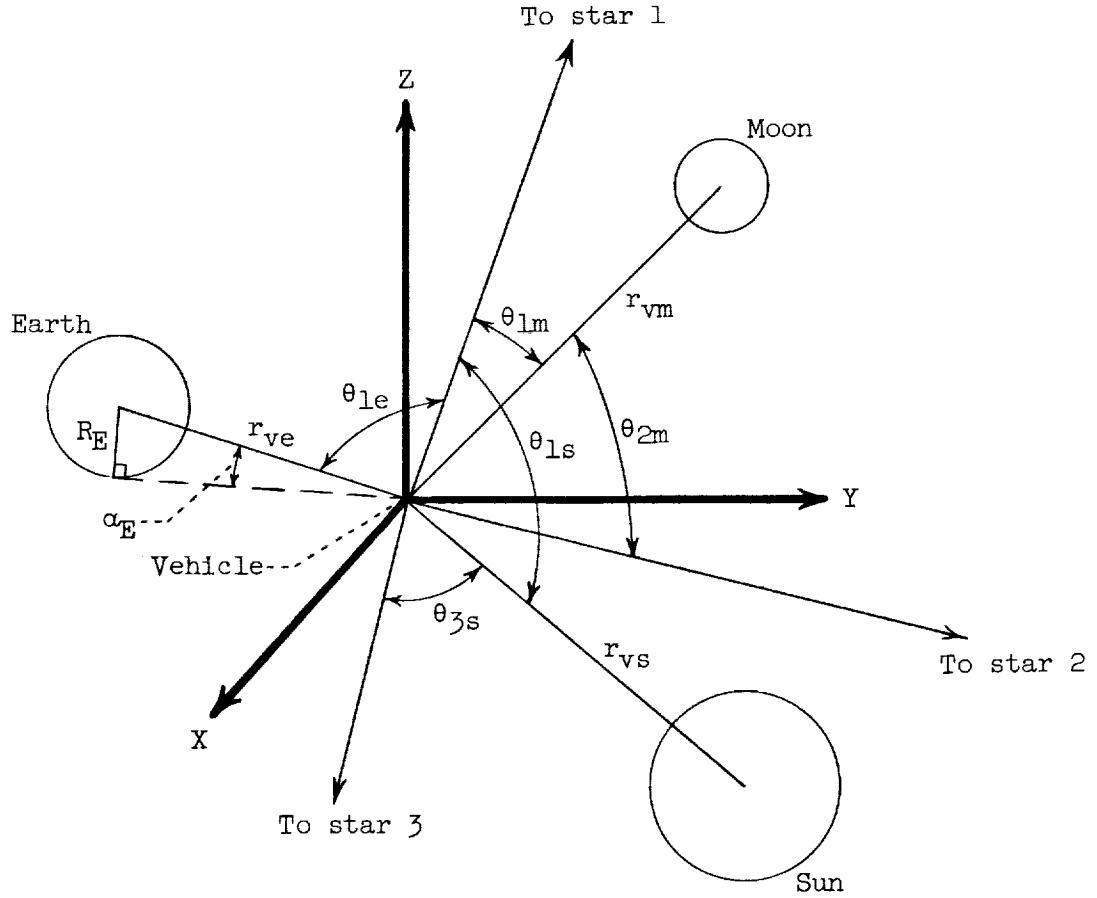
$$r_{vm} \cos \theta_{2m} = l_2 x_{vm} + m_2 y_{vm} + n_2 z_{vm} \quad (36e)$$

$$r_{vs} \cos \theta_{3s} = l_3 x_{vs} + m_3 y_{vs} + n_3 z_{vs} \quad (36f)$$

Elimination of  $r_{ve}$ ,  $r_{vm}$ , and  $r_{vs}$  between appropriate pairs of equations and substitution of equations (34) into the resulting equations leads to the following solution for determination of vehicle position by method XI:

$$\begin{Bmatrix} x_{ve} \\ y_{ve} \\ z_{ve} \end{Bmatrix} = \begin{bmatrix} l_1 - l_2 \frac{\cos \theta_{1e}}{\cos \theta_{2e}} & m_1 - m_2 \frac{\cos \theta_{1e}}{\cos \theta_{2e}} & n_1 - n_2 \frac{\cos \theta_{1e}}{\cos \theta_{2e}} \\ l_1 - l_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} & m_1 - m_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} & n_1 - n_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} \\ l_1 - l_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} & m_1 - m_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} & n_1 - n_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} \end{bmatrix}^{-1} \times \begin{Bmatrix} 0 \\ \left( l_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} - l_1 \right) x_{em} + \left( m_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} - m_1 \right) y_{em} + \left( n_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} - n_1 \right) z_{em} \\ \left( l_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} - l_1 \right) x_{es} + \left( m_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} - m_1 \right) y_{es} + \left( n_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} - n_1 \right) z_{es} \end{Bmatrix} \quad (37)$$

Method XII.- In method XII, measurements are made of included angles between star and the centers of the earth, moon, and sun, between another star and the moon center, between a third star and the sun center, and the angular diameter of the earth. (See sketch 12.)



Sketch 12

The cosines of the angles included between star 1 and the earth center, between stars 1 and 2 and the moon center, and between stars 1 and 3 and the sun center are expressed, respectively, by the following equations:

$$R_E \frac{\cos \theta_{1e}}{\sin \alpha_E} = l_1 x_{ve} + m_1 y_{ve} + n_1 z_{ve} \quad (38)$$

$$r_{vm} \cos \theta_{1m} = l_1 x_{vm} + m_1 y_{vm} + n_1 z_{vm} \quad (38)$$

$$r_{vm} \cos \theta_{2m} = l_2 x_{vm} + m_2 y_{vm} + n_2 z_{vm} \quad (38)$$

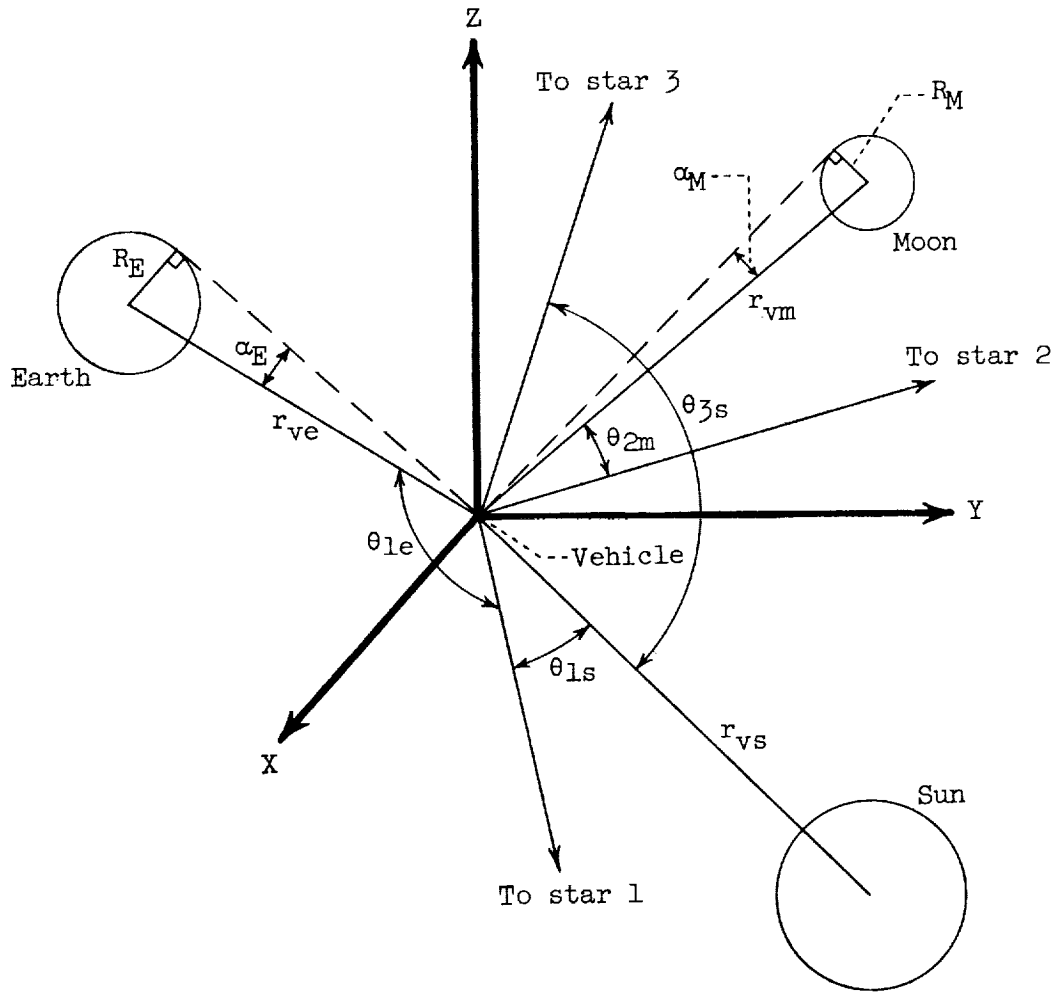
$$r_{vs} \cos \theta_{1s} = l_1 x_{vs} + m_1 y_{vs} + n_1 z_{vs} \quad (38)$$

$$r_{vs} \cos \theta_{3s} = l_3 x_{vs} + m_3 y_{vs} + n_3 z_{vs} \quad (38)$$

After elimination of  $r_{vm}$  and  $r_{vs}$  between appropriate pairs of equations substitution of equations (34) into the resulting equations, the solution for Method XII is as follows:

$$\begin{Bmatrix} r_e \\ r_e \\ r_e \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_1 - l_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} & m_1 - m_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} & n_1 - n_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} \\ l_1 - l_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} & m_1 - m_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} & n_1 - n_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} \end{bmatrix}^{-1} \times \begin{Bmatrix} R_E \frac{\cos \theta_{1e}}{\sin \alpha_E} \\ \left( l_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} - l_1 \right) x_{em} + \left( m_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} - m_1 \right) y_{em} + \left( n_2 \frac{\cos \theta_{1m}}{\cos \theta_{2m}} - n_1 \right) z_{em} \\ \left( l_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} - l_1 \right) x_{es} + \left( m_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} - m_1 \right) y_{es} + \left( n_3 \frac{\cos \theta_{1s}}{\cos \theta_{3s}} - n_1 \right) z_{es} \end{Bmatrix} \quad (39)$$

Method XIII.— Method XIII involves measurements of included angles between one of two stars and the sun center, between one of these stars and the earth center, between another star and the moon center, and the angular diameters of the sun and moon. (See sketch 13.)



Sketch 13

The cosines of the angles included by star 1 and the earth center, star 2 and the moon center, stars 1 and 3 and the sun center are given, respectively, by the following equations:

$$R_E \frac{\cos \theta_{1e}}{\sin \alpha_E} = l_1 x_{ve} + m_1 y_{ve} + n_1 z_{ve} \quad (40a)$$

$$R_M \frac{\cos \theta_{2m}}{\sin \alpha_M} = l_2 x_{vm} + m_2 y_{vm} + n_2 z_{vm} \quad (40b)$$

$$r_{VS} \cos \theta_{1S} = l_1 x_{VS} + m_1 y_{VS} + n_1 z_{VS} \quad (40c)$$

$$r_{VS} \cos \theta_{3S} = l_3 x_{VS} + m_3 y_{VS} + n_3 z_{VS} \quad (40d)$$

Elimination of  $r_{VS}$  between equations (40c) and (40d) and substitution of equations (34) into the result yields the following solution for determination of icle position by method XIII:

$$\begin{Bmatrix} e \\ e \\ e \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_1 - l_3 \frac{\cos \theta_{1S}}{\cos \theta_{3S}} & m_1 - m_3 \frac{\cos \theta_{1S}}{\cos \theta_{3S}} & n_1 - n_3 \frac{\cos \theta_{1S}}{\cos \theta_{3S}} \end{bmatrix}^{-1} \times \left\{ \begin{array}{l} R_E \frac{\cos \theta_{1e}}{\sin \alpha_E} \\ R_M \frac{\cos \theta_{2m}}{\sin \alpha_M} - (l_2 x_{em} + m_2 y_{em} + n_2 z_{em}) \\ \left( l_3 \frac{\cos \theta_{1S}}{\cos \theta_{3S}} - l_1 \right) x_{es} + \left( m_3 \frac{\cos \theta_{1S}}{\cos \theta_{3S}} - m_1 \right) y_{es} + \left( n_3 \frac{\cos \theta_{1S}}{\cos \theta_{3S}} - n_1 \right) z_{es} \end{array} \right\} \quad (41)$$

#### CONCLUDING REMARKS

Equations have been developed for 13 different combinations of simultaneous board optical angular measurements which result in a position fix in earth-moon ce. The various combinations were selected to give a cross section of the ge number of methods available. The equations pertinent to the various methods e been presented mainly as background material for studies now underway to ermine practical and accurate methods of onboard navigation in different ions of earth-moon space. The equations will be especially useful for detailed lies of any method, such as a complete error analysis of a particular system optical measurements.

No consideration has been given to the instrumentation difficulties, the main purpose of the report being to determine the basic equations pertinent to the navigational problem. Each of the methods includes the minimum number of measurements for a nonredundant mathematical solution. For a practical onboard navigational system, the number of measurements needed for any method of determining the vehicle position vector can be reduced by augmenting the system with advance information. For example, the number of star-to-body angle measurements can be reduced from three to two if the approximate location of the trajectory is generally known at any time throughout the trip.

For a practical application of any of the methods, the measurements would require some corrections for the effects of aberration, refraction, and other factors. In addition, the accuracy of any method would not only depend upon the accuracy of the sighting instruments but on an accurate knowledge of the body diameters, distance between the earth, moon, and sun, and exact positions of the stars.

Langley Research Center,  
National Aeronautics and Space Administration,  
Langley Station, Hampton, Va., November 19, 1962.

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